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# Uniform Hausdorff Strong Uniqueness

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For a linear subspace $M$ of a normed linear space $X$ and $x \in X$, let $P_M(x)$ be the set of all best approximations to $x$ from $M$. We study the subspaces $M$ such that $P_M$ is uniformly Hausdorff strongly unique. The $1\frac{1}{2}$-ball property implies uniform Hausdorff strong uniqueness, but the converse is false. We obtain that, if $P_M$ is uniformly Hausdorff strongly unique, then $P_M$ is Lipschitz continuous. When $M$ is a hyperplane, $P_M$ is Hausdorff strongly unique for some $x \in X \backslash M$ if and only if $P_M$ is uniformly Hausdorff strongly unique. In $C[a, b]$, $P_M$ is uniformly Hausdorff strongly unique if and only if $M$ is one dimensional and Chebyshev. 

Let $X$ be a normed linear space, and for $x \in X$ and $r \geqq 0$ denote

$$B(x, r) = B_X(x, r) = \{y \in X: \|y - z\| \leqq r\}.$$

For a nonempty subset $M$ of $X$ and each $x \in X$ we denote by $P_M(x)$ the set of all best approximations to $x$ from $M$, i.e.,

$$P_M(x) = \{m_0 \in M: \|x - m_0\| = d(x, M)\}.$$

The set $M$ is called:

(1) *proximinal* in $X$ if, for each $x \in X$, $P_M(x)$ is nonempty.

(2) *Chebyshev* in $X$ if, for each $x \in X$, $P_M(x)$ is a singleton.

Throughout this article, unless otherwise specified, $M$ will denote a linear (not necessarily closed) subspace of $X$.

For $x \in X$ and $\varepsilon \geqq 0$, we denote by $P_M^\varepsilon(x)$ the set of all $\varepsilon$-approximations to $x$ from $M$, i.e.,

$$P_M^\varepsilon(x) = \{m_0 \in M: \|x - m_0\| \leqq d(x, M) + \varepsilon\}.$$

Notice that $P_M^0(x) = P_M(x)$. Clearly, for each $\varepsilon \geqq 0$, we have

$$P_M^\varepsilon(x) = M \cap B(x, d(x, M) + \varepsilon)$$

and for each $\varepsilon > 0$, $P_M^\varepsilon(x) \neq \varnothing$.

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For a set $A \subset X$ and $\varepsilon \geqq 0$, the closure of the $\varepsilon$-neighborhood of $A$, denoted by $A_\varepsilon$, is

$$A_\varepsilon = \overline{B_\varepsilon(A)} = \{x \in X: d(x, A) \leqq \varepsilon\}.$$

Using the convention that $d(x, \phi) = \infty$, it follows that, for $A = \phi$, we have $A_\varepsilon = \phi$ for each (finite) $\varepsilon \geqq 0$.

Many mathematicians have studied strong uniqueness when $M$ is Chebyshev. W. Li [6] defined and studied Hausdorff strong uniqueness in $C(T)$ when $M$ is proximinal. Here we will give the definition of Hausdorff strong uniqueness and define uniform Hausdorff strong uniqueness for any normed space $X$. In this article we give a characterization of (uniform) Hausdorff strong uniqueness. We show that the $1\frac{1}{2}$-ball property is strictly stronger than uniform Hausdorff strong uniqueness. But if $P_M$ is uniformly Hausdorff strongly unique then $P_M$ is Lipschitz continuous. In $C[a, b]$, we characterize a subspace whose metric projection is uniformly Hausdorff strongly unique. Finally we show that for a hyperplane $M$, $P_M$ is Hausdorff strongly unique for some $x \in X \backslash M$ if and only if $P_M$ is uniformly Hausdorff strongly unique.

Definition 1 [6]. Let $M$ be a proximinal closed subspace of $X$ and let $x \in X$. The set $P_M(x)$ is said to be *Hausdorff strongly unique* if

$$r(x) := \inf\left\{\frac{\|x - m\| - d(x, M)}{d(m, P_M(x))} : m \in M \backslash P_M(x)\right\} > 0. \tag{1.1}$$

If $P_M(x)$ is Hausdorff strongly unique, it follows that

$$\|x - m\| \geqq d(x, M) + r(x)\, d(m, P_M(x)) \tag{1.2}$$

for each $m \in M$. In addition, $r = r(x)$ is the largest constant for which (1.2) holds for each $m \in M$.

Notice that if $M$ is Chebyshev, then this condition reduces to the well-known condition that $P_M(x)$ is strongly unique;

$$\|x - m\| \geqq \|x - P_M(x)\| + r(x)\|m - P_M(x)\|,$$

for each $m \in M$.

Lemma 2. *Let $M$ be a proximinal subspace of a normed linear space $X$. Then*

(1) *for any $x, y \in X$, $\|x - y\| \leqq d(x, M) + d(y, P_M(x))$.*

(2) *for each $x \in X$, $\|x - m\| \leqq d(x, M) + d(m, P_M(x))$ for any $m \in M$.*

(3) *if* $P_M(x)$ *is Hausdorff strongly unique, then* $r(x) \leqq 1$.

(4) $P_M(x)$ *is Hausdorff strongly unique with* $r(x) = 1$ *if and only if*

$$\|x - m\| = d(x, M) + d(m, P_M(x))$$

*for each* $m \in M$.

*Proof.* (1) Let $x, y \in X$ be given. Then for each $m \in P_M(x)$,

$$\|x - y\| \leqq \|x - m\| + \|m - y\| = d(x, M) + \|m - y\|.$$

Thus $\|x - y\| \leqq d(x, M) + d(y, P_M(x))$.

(2) Take $y = m$ in (1).

(3) Assume that $r(x) > 1$ and $m \notin P_M(x)$. By (2),

$$\begin{aligned} \|x - m\| &\geqq d(x, M) + r(x)\, d(m, P_M(x)) \\ &> d(x, M) + d(m, P_M(x)) \\ &\geqq \|x - m\|. \end{aligned}$$

This is a contradiction. Thus $r(x) \leqq 1$.

(4) If $P_M(x)$ is Hausdorff strongly unique with $r(x) = 1$, then

$$\|x - m\| \geqq d(x, M) + d(m, P_M(x))$$

for each $m \in M$. But (2) implies the reverse inequality. Thus $\|x - m\| = d(x, M) + d(m, P_M(x))$ for each $m \in M$.

Conversely, if $\|x - m\| = d(x, M) + d(m, P_M(x))$ for each $m \in M$, then $P_M(x)$ is Hausdorff strongly unique with $r(x) \geqq 1$. By (3), $r(x) = 1$.

THEOREM 3. *Let* $M$ *be a proximinal subspace of a normed linear space* $X$, $x \in X$, *and* $r > 0$. *Then the following statements are equivalent:*

(1) $P_M(x)$ *is Hausdorff strongly unique with* $r \leqq r(x)$;

(2) $P_M^\varepsilon(x) \subset P_M(x)_{\varepsilon/r}$ *for each* $\varepsilon \geqq 0$.

*Proof.* (1) $\Rightarrow$ (2) Suppose (1) holds. Then $\|x - m\| \geqq d(x, M) + rd(m, P_M(x))$ for each $m \in M$. Let $\varepsilon \geqq 0$ and $m \in P_M^\varepsilon(x)$. Then we have

$$rd(m, P_M(x)) + d(x, M) \leqq \|x - m\| \leqq d(x, M) + \varepsilon.$$

So $d(m, P_M(x)) \leqq \varepsilon/r$. Thus $m \in P_M(x)_{\varepsilon/r}$. Therefore, $P_M^\varepsilon(x) \subset P_M(x)_{\varepsilon/r}$ for each $\varepsilon \geqq 0$.

(2) ⇒ (1) Suppose (2) holds. Let $m \in M$ and $\varepsilon = \|x-m\| - d(x, M)$. Then $\varepsilon \geqq 0$ and $m \in P_M^\varepsilon(x)$, so $m \in P_M(x)_{\varepsilon/r}$. Thus

$$d(m, P_M(x)) \leqq \frac{\varepsilon}{r} = \frac{1}{r}(\|x-m\| - d(x, M)),$$

i.e.,

$$rd(m, P_M(x)) + d(x, M) \leqq \|x-m\|$$

for each $m \in M$. Therefore $P_M(x)$ is Hausdorff strongly unique with $r \leqq r(x)$.

Definition 4. Let $M$ be a proximinal subspace of $X$. The metric projection $P_M$ is said to be *uniformly Hausdorff strongly unique* if

$$r(M) := \inf\{r(x) : x \in X\} > 0.$$

Note that $r(M)$ is the largest number so that

$$\|x-m\| \geqq d(x, M) + r(M)\, d(m, P_M(x)) \tag{3.1}$$

for each $x \in X$ and $m \in M$. Moreover, if $r(M) = 1$, then $r(x) = 1$ for all $x \in X$.

Example 5. [There is a proximinal subspace $M$ which is not Chebyshev so that $P_M$ is uniformly Hausdorff strongly unique.] Let $M = \text{span}\{(1, 0)\}$ be a subspace of $X = \mathbb{R}^2$ with the norm: $\|x\| = \|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$. Then $M$ is proximinal (not Chebyshev) and $P_M$ is uniformly Hausdorff strongly unique with $r(M) = 1$. Clearly for each $x = (x_1, x_2) \in X$,

$$P_M(x) = \{(\alpha, 0) : \alpha \in [x_1 - |x_2|, x_1 + |x_2|]\}$$

and $d(x, M) = |x_2|$.

Now we want to prove that $P_M$ is uniformly Hausdorff strongly unique with $r(M) = 1$. Let $x = (x_1, x_2) \in X$ and $m = (a, 0) \in M$ be fixed. Then $\|x-m\| = \max\{|x_1 - a|, |x_2|\}$.

If $\|x-m\| = |x_2|$, then $m \in P_M(x)$ so $d(m, P_M(x)) = 0$ and $d(x, M) = |x_2|$. Thus $\|x-m\| = d(x, M) + d(m, P_M(x))$.

If $\|x-m\| = |x_1 - a|$, then $m \notin P_M(x)$, so either $a > x_1 + |x_2|$ or $a < x_1 - |x_2|$. If $a > x_1 + |x_2|$, then $d(m, P_M(x)) = a - x_1 - |x_2|$, so $d(x, M) + d(m, P_M(x)) = |x_2| + a - x_1 - |x_2| = a - x_1 = \|x-m\|$. If $a < x_1 - |x_2|$, then $d(m, P_M(x)) = x_1 - |x_2| - a$, so $d(x, M) + d(m, P_M(x)) = |x_2| + x_1 - |x_2| - a = x_1 - a = \|x-m\|$.

Therefore for each $x \in X$, $\|x-m\| = d(x, M) + d(m, P_M(x))$ for any $m \in M$, Thus $P_M$ is uniformly Hausdorff strongly unique with $r(M) = 1$.

As an immediate consequence of Theorem 3, we obtain

COROLLARY 6. *Let $M$ be a proximinal subspace of a normed linear space $X$ and $0 < r$. The following statements are equivalent.*

(1) *$P_M$ is uniformly Hausdorff strongly unique with $r \leqq r(M)$;*

(2) *For each $x \in X$,*

$$P_M^{\varepsilon}(x) \subset P_M(x)_{\varepsilon/r}$$

*for each $\varepsilon \geqq 0$.*

When $M$ is proximinal in $X$, $P_M$ is uniformly Hausdorff strongly unique for $P_M$ with $r(M) = 1$ is equivalent to what G. Godini [5] called property (*) of $M$.

DEFINITION 7 [5]. A subspace $M$ of a normed linear space $X$ has *property* (*) in $X$ if for each $x \in X$ with $P_M(x) \neq \varnothing$ and each $m \in M$ we have that

$$d(m, P_M(x)) = \|x - m\| - d(x, M). \tag{7.1}$$

DEFINITION 8 [11]. A subspace $M$ of a normed linear space $X$ has the *$1\frac{1}{2}$-ball property* in $X$ if the conditions $m \in M$, $x \in X$, $r_i \geqq 0$ $(i = 1, 2)$, $M \cap B(x, r_2) \neq \varnothing$, and $\|x - m\| < r_1 + r_2$ imply that

$$M \cap B(m, r_1) \cap B(x, r_2) \neq \varnothing.$$

THEOREM 9. *Let $M$ be a proximinal subspace of a normed linear space $X$. The following statements are equivalent.*

(1) *$P_M$ is uniformly Hausdorff strongly unique with $r(M) = 1$;*

(2) *$M$ has property (*) in $X$;*

(3) *For each $x \in X$.*

$$P_M^{\varepsilon}(x) = P_M(x)_{\varepsilon} \cap M$$

*for any $\varepsilon \geqq 0$;*

(4) *$M$ has the $1\frac{1}{2}$-ball property in $X$.*

*Proof.* The equivalence of statements (2), (3), and (4) was proven by G. Godini [5].

(1) $\Leftrightarrow$ (2) Clearly, $P_M$ is uniformly Hausdorff strongly unique with $r(M) = 1$ if and only if $r(x) = 1$ for each $x \in X$. By (4) of Lemma 2, this is equivalent to $M$ having property (*).

Recall [8] that a subspace $M$ is a "semi-$L$-summand" in $X$ if $M$ is Chebyshev and for each $x \in X$

$$\|x\| = \|P_M(x)\| + \|x - P_M(x)\|.$$

COROLLARY 10. *If $M$ is a semi-$L$-summand in $X$, then $P_M$ is uniformly Hausdorff strongly unique with $r(M) = 1$.*

*Proof.* Since $P_M$ is "additive modulo $M$" (i.e., $P_M(x+m) = P_M(x) + m$ for each $x \in X$ and $m \in M$), by replacing $x$ by $x - m$ in the definition of semi-$L$-summand we obtain that

$$\begin{aligned}\|x-m\| &= \|P_M(x-m)\| + \|x - m - P_M(x-m)\| \\ &= \|P_M(x) - m\| + \|x - P_M(x)\| \\ &= d(m, P_M(x)) + d(x, M)\end{aligned}$$

for each $m \in M$. Thus $M$ has property (*). By Theorem 9, the result follows.

*Remark.* By Theorem 9, the $1\frac{1}{2}$-ball property implies uniformly Hausdorff strongly unique. But the converse is not true in general.

EXAMPLE 11. [There is a subspace $M$ for which $P_M$ is uniformly Hausdorff strongly unique with $0 < r(M) < 1$ but $M$ fails the $1\frac{1}{2}$-ball property.] In $C[0, 1]$, let $0 < r < 1$ and $M = \text{span}\{m\}$ where

$$m_0(t) = (r-1)\, t + 1$$

for any $t \in [0, 1]$ and $0 < r < 1$. Clearly $\|m_0\| = 1$. Then $M$ is a Chebyshev subspace of $C[0, 1]$. Fix any $f \in C[0, 1]$. By the alternation theorem, there exist $t_0$, $t_1 \in [0, 1]$ such that

$$\begin{aligned}f(t_0) - \alpha_f m_0(t_0) &= \|f - \alpha_f m_0\| \\ f(t_1) - \alpha_f m_0(t_1) &= -\|f - \alpha_f m_0\|\end{aligned}$$

or

$$\begin{aligned}f(t_0) - \alpha_f m_0(t_0) &= -\|f - \alpha_f m_0\| \\ f(t_1) - \alpha_f m_0(t_1) &= \|f - \alpha_f m_0\|,\end{aligned}$$

where $P_M(f) = \{\alpha_f m_0\}$. We may assume

$$\begin{aligned}f(t_0) - \alpha_f m_0(t_0) &= \|f - \alpha_f m_0\| \\ f(t_1) - \alpha_f m_0(t_1) &= -\|f - \alpha_f m_0\|.\end{aligned}$$

Note that $r=\min\{|m_0(t)|: t\in[0,1]\}$. Let $\alpha m_0\in M$ be given. If $\alpha_f\geqq\alpha$, then

$$\begin{aligned}\|f-\alpha m_0\| &= \|f-\alpha_f m_0+(\alpha_f-\alpha)\, m_0\|\\ &\geqq |f(t_0)-\alpha_f m_0(t_0)+(\alpha_f-\alpha)\, m_0(t_0)|\\ &\geqq \|f-\alpha_f m_0\|+|\alpha_f-\alpha|\min_{0\leqq t\leqq 1}|m_0(t)|\\ &= \|f-\alpha_f m_0\|+\frac{r}{\|m_0\|}\|\alpha_f m_0-\alpha m_0\|.\end{aligned}$$

If $\alpha_f<\alpha$, then

$$\begin{aligned}\|f-\alpha m_0\| &= \|f-\alpha_f m_0+(\alpha_f-\alpha)\, m_0\|\\ &\geqq |f(t_1)-\alpha_f m_0(t_1)+(\alpha_f-\alpha)\, m_0(t_1)|\\ &\geqq \|f-\alpha_f m_0\|+|\alpha_f-\alpha|\min_{0\leqq t\leqq 1}|m_0(t)|\\ &= \|f-\alpha_f m_0\|+\frac{r}{\|m_0\|}\|\alpha_f m_0-\alpha m_0\|.\end{aligned}$$

Since $\|m_0\|=1$ and $P_M(f)=\{\alpha_f m_0\}$,

$$\|f-m\|\geqq d(f, M)+rd(m, P_M(f))$$

for any $m\in M$. Since $f$ was arbitrary, $P_M$ is uniformly Hausdorff strongly unique and $r\leqq r(M)$. Now we want to prove that $r$ is the largest number and hence $r=r(M)$. Define $f(t)=(1+r)/2$ for any $t\in[0,1]$. Then

$$\left\|f-\frac{1}{2}m_0\right\|=\max\left\{\left|\frac{1+r}{2}-\frac{1}{2}\right|, \left|\frac{1+r}{2}-\frac{r}{2}\right|\right\}=\frac{1}{2}$$

since $P_M(f)=\{m_0\}$. Thus

$$\|f-\tfrac{1}{2}m_0\|=d(f, M)+rd(\tfrac{1}{2}m_0, P_M(f)).$$

Therefore $r$ is the largest number. We have shown that $P_M$ is uniform Hausdorff strongly unique with $0<r(M)<1$. But in [10] we observed that the only finite-dimensional subspace of $C[0,1]$ which has the $1\frac{1}{2}$-ball property is $M_1=\operatorname{span}\{1\}$. Thus $M$ fails the $1\frac{1}{2}$-ball property.

Thus, in general, uniformly Hausdorff strongly unique does not imply the $1\frac{1}{2}$-ball property. So uniform Hausdorff strong uniqueness is strictly weaker than the $1\frac{1}{2}$-ball property. But we still have the following property: If $P_M$ is uniformly Hausdorff strongly unique, then $P_M$ is Lipschitz continuous.

Let $X$ be a normed linear space and $H(X)$ denote the family of all non-empty closed, bounded, convex subsets of $X$.

Define $h: H(X) \times H(X) \to \mathbb{R}$ by

$$h(A, B) = \sup_{a \in A} d(a, B),$$

where $d(a, B) = \inf_{b \in B} \|a - b\|$. The *Hausdorff metric* on $H(X)$ is defined by

$$H(A, B) = \max\{h(A, B), h(B, A)\}.$$

Recall that $P_M$ is *pointwise Lipschitz u.H.s.c.* at $x_0$ if there exists $\lambda_{x_0} > 0$ such that for each $x \in X$,

$$h(P_M(x), P_M(x_0)) \leqq \lambda_{x_0} \|x - x_0\|.$$

THEOREM 12. *Let $M$ be a subspace of a normed linear space $X$. If $P_M(x)$ is Hausdorff strongly unique, then $P_M$ is pointwise Lipschitz u.H.s.c. at $x$ with constant $2/r(x)$.*

*Proof.* Suppose that $\|x - m\| \geqq d(x, M) + r(x)\, d(m, P_M(x))$ for any $m \in M$, Let $y \in X$ and $m \in P_M(y)$ be given. Then

$$\begin{aligned} r(x)\, d(m, P_M(x)) &\leqq \|x - m\| - d(x, M) \\ &\leqq \|x - y\| + \|y - m\| - d(x, M) \\ &= \|x - y\| + d(y, M) - d(x, M) \\ &\leqq 2\|x - y\|. \end{aligned}$$

Thus $h(P_M(y), P_M(x)) \leqq (2/r(x))\|x - y\|$. Therefore $P_M$ is pointwise Lipschitz u.H.s.c. at $x$ with constant $2/r(x)$.

DEFINITION 13. Let $M$ be a proximinal subspace of a normed linear space $X$. $P_M$ is said to be *Lipschitz continuous* with constant $\lambda$ if

$$H(P_M(x), P_M(y)) \leqq \lambda \|x - y\|$$

for any $x, y \in X$.

THEOREM 14. *Let $M$ be a proximinal subspace of $X$. If $P_M$ is uniformly Hausdorff strongly unique, then $P_M$ is Lipschitz continuous with Lipschitz constant $2/r(M)$.*

*Proof.* Let $r = r(M)$. Since $P_M$ is uniformly strongly unique, for each $x \in X$ and $m \in M$.

$$\|x - m\| \geqq d(x, M) + rd(m, P_M(x)).$$

Let $x, y \in X$ and $m \in P_M(y)$. Then

$$\begin{aligned} rd(m, P_M(x)) &\leqq \|x-m\| - d(x, M) \\ &\leqq \|x-y\| + \|y-m\| - d(x, M) \\ &= \|x-y\| + d(y, M) - d(x, M) \\ &\leqq 2\ \|x-y\|. \end{aligned}$$

Thus $d(m, P_M(x)) \leqq (2/r)\ \|x-y\|$. Since $m$ was arbitrary in $P_M(y)$, $h(P_M(y),\ M_M(x)) \leqq (2/r)\ \|x-y\|$. By symmetry, $h(P_M(x),\ P_M(y)) \leqq (2/r)\ \|x-y\|$. Therefore $H(P_M(x), P_M(Y)) \leqq (2/r)\ \|x-y\|$, and $P_M$ is Lipschitz continuous with Lipschitz constant $2/r$.

THEOREM 15. *Let $M$ be a finite-dimensional subspace of $C[a, b]$. The following statements are equivalent.*

(1) *$P_M$ is uniformly Hausdorff strongly unique;*

(2) *$M$ is one dimensional and Chebyshev.*

*Proof.* (1)$\Rightarrow$(2) Suppose (1) holds. By Theorem 14, $P_M$ is Lipschitz continuous. Since $[a, b]$ is a connected compact Hausdorff space, $M$ is Chebyshev [2]. By a result of Cline [3], $M$ is one dimensional. Thus (2) holds.

(2)$\Rightarrow$(1) Suppose $M$ is one dimensional and Chebyshev. Then there exists $x_1 \in C[a, b]$ such that $x_1$ has no zero and $M = \operatorname{span}\{x_1\}$. By the same argument to Example 11, we can prove that $P_M$ is uniformly Hausdorff strongly unique.

*Remark.* In $C(T)$, where $T$ is a connected compact Hausdorff space, the above theorem is also true. In fact, (1)$\Rightarrow$(2) is the same proof as the above and by a Theorem in [9] and the same argument to Example 11, (2)$\Rightarrow$(1). (2)$\Rightarrow$(1) also follows from the result of Wu Li [7].

In general, the converse of Theorem 14 is not true. We have an example.

EXAMPLE 16. [$P_M$ is Lipschitz continuous, but $P_M$ is not uniformly Hausdorff strongly unique.] Let $X = \mathbb{R}^2$ have the norm

$$\|x\| = \|(x(1), x(2))\| = \{|x(1)|^2 + |x(2)|^2\}^{1/2}.$$

Let $M = \operatorname{span}\{(1, 0)\}$. Then clearly $M$ is Chebyshev and $P_M$ is Lipschitz continuous, i.e., $\|P_M(x) - P_M(y)\| \leqq \|x-y\|$. Suppose that $P_M$ is uniformly Hausdorff strongly unique and let $r = r(M)$. Let $x = (0, 1)$ so $P_M(x) = (0, 0)$ and $d(x, M) = 1$. If $r = 1$, let $m = (1, 0)$. Then

$$d(x, M) + rd(m, P_M(x)) = 2 > \|x-m\|$$

which contradicts uniform Hausdorff strong uniqueness. If $0<r<1$, let $\alpha=r/(1-r^2)$ and $m=(\alpha, 0)$. Then

$$d(x, M)+rd(m, P_M(x))=1+r\sqrt{1+\alpha^2}<\sqrt{1+\alpha^2}=\|x-m\|$$

which also contradicts uniform Hausdorff strong uniqueness. Thus $P_M$ is not uniformly Hausdorff strongly unique.

COROLLARY 17 [11]. *Let $M$ be a proximinal subspace of a Banach space $X$. If $M$ has the $1\frac{1}{2}$-ball property in $X$, then $P_M$ is Lipschitz continuous with Lipschitz constant* 2.

*Proof.* By Theorem 9, $P_M$ is uniformly Hausdorff strongly unique with $r(M)=1$. The result now follows by Theorem 14.

THEOREM 18 [4]. *Let $M$ be a finite-dimensional subspace of $X$. If $P_M$ is Lipschitz continuous, then $P_M$ has a Lipschitz continuous selection which is homogeneous and additive modulo $M$.*

COROLLARY 19. *Let $M$ be an n-dimensional subspace of $X$. If $P_M$ is uniformly Hausdorff strongly unique, then $P_M$ has a Lipschitz continuous selection which is homogeneous and additive modulo $M$. Moreover, if $M$ has the $1\frac{1}{2}$-ball property in $X$, then $P_M$ has a Lipschitz continuous selection which is homogeneous and additive modulo $M$.*

*Proof.* If $M$ has the $1\frac{1}{2}$-ball property, then by Theorem 9, $P_M$ is uniformly Hausdorff strongly unique. By Theorem 14, such maps are Lipschitz continuous. The result now follows by Theorem 18.

*Remark.* M. W. Bartelt and H. W. McLaughlin [1] proved that if the best approximation to $x$ from $M$ is strongly unique, then so is the best approximation to every element of span$\{M, x\}$. This result can be generalized.

THEOREM 20. *Let $M$ be a subspace of $X$ and $x\in X\backslash M$. If $P_M(x)$ is Hausdorff strongly unique, then $P_M$ is uniformly Hausdorff strongly unique in span$\{M, x\}$.*

*Proof.* Suppose $P_M(x)$ is Hausdorff strongly unique, i.e., there exists $r>0$ such that $\|x-m\|\geqq d(x, M)+rd(m, P_M(x))$ for any $m\in M$. The case $M=\{0\}$ is trivial. Assume that $M\neq\{0\}$. It suffices to show that for any scalar $a$ and $m_0\in M$,

$$\|(ax+m_0)-m\|\geqq d(ax+m_0, M)+rd(m, P_M(ax+m_0))$$

for any $m \in M$. Let $ax+m_0 \in \text{span}\{M, x\}$. If $a=0$, it is obvious. If $a \neq 0$, then for all $m \in M$,

$$\begin{aligned}\|ax+m_0-m\| &= |a| \left\| x - \frac{1}{a}(m-m_0) \right\| \\ &\geqq |a| \left\{ d(x, M) + rd\left( \frac{1}{a}(m-m_0), P_M(x) \right) \right\} \\ &= d(ax+m_0, M) + rd(m-m_0, P_M(ax)) \\ &= d(ax+m_0, M) + rd(m, P_M(ax+m_0)).\end{aligned}$$

Since $ax+m_0$ was arbitrary in $\text{span}\{M, x\}$, $P_M$ is uniformly Hausdorff strongly unique.

COROLLARY 21. *Let $M$ be a hyperplane in $X$.*

*The following statements are equivalent.*

(1) *There exists $x \in X \backslash M$ such that $P_M(x)$ is Hausdorff strongly unique*;

(2) *$P_M$ is uniformly Hausdorff strongly unique in $X$.*

*Proof.* (2) ⇒ (1) is obvious.

(2) ⇒ (1) Suppose that (1) holds. By Theorem 20, $P_M$ is uniformly Hausdorff strongly unique in $\text{span}\{M, x\}$. Since $\text{span}\{M, x\} = X$, $P_M$ is uniformly Hausdorff strongly unique in $X$.

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